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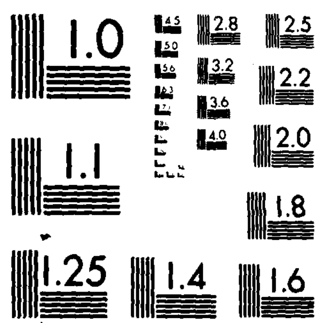

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(6) The Fluid Dynamic Limit of the Nonlinear Boltzmann Equation*

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Abstract

Solutions of the nonlinear Boltzmann equation are constructed up to the first appearance of shocks in the corresponding fluid dynamics. This construction assumes the knowledge of solutions of the Euler equations for compressible gas flow. The Boltzmann solution is found as a truncated Hilbert expansion with a remainder, and the remainder term solves a weakly nonlinear equation which is solved by iteration. The solutions found have special initial values. They should serve as "outer expansions" to which initial layers, boundary layers and shock layers can be matched.

Introduction

The Boltzmann equation of kinetic theory gives a statistical description of a gas of interacting particles. An important property of this equation is its asymptotic equivalence to the Euler or Navier–Stokes equations of compressible gas dynamics, in the limit of small mean free path. We propose that, as one aspect of this asymptotic relationship, the question of existence of solutions of the Boltzmann equation can be reduced to the existence problem for the gas dynamic equations. Although the latter problem has received only partial solutions, the gas dynamic equations are much simpler than the Boltzmann equation and have been studied extensively.

This paper makes a first step in that reduction of the existence question by showing that any smooth solution of the Euler equations can be used to make a corresponding solution of the Boltzmann equation. It does not address the difficulties of initial values, shocks, and boundaries. The solution produced here comes from a truncated asymptotic expansion and has special initial values, is periodic in space, and is valid only until the first appearance of shocks. It can serve as an "outer solution" to which special solutions, in initial layers, boundary layers, and shock layers, can be matched.

However, the analysis of these layers is not complete. Grad has given a formal treatment of the initial layer matching problem in [6]. It requires solving the spatially homogeneous nonlinear Boltzmann equation and showing that the solution approaches a Maxwellian asymptotically in time, for any initial data. Arkeryd [1] has shown that this is true, but with only weak L^1

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convergence, which is not strong enough for the completion of the expansion. The profile of a weak, steady shock was analyzed by Nicolaenko [11] for the Boltzmann equation for hard spheres. The effects of time dependence must also be included for the shock layer analysis. For boundary layer problems, one must solve the steady nonlinear Boltzmann equation in a half-space and show approach to a Maxwellian at infinity. Guiraud has analyzed a more general weakly nonlinear problem in [9].

The results presented here are valid for a physically interesting time period, until the first occurrence of shocks, and concern solutions which are far from spatial equilibrium, so that the corresponding gas dynamics is strongly nonlinear. Previously, Glikson [4] and Kaniel and Shinbrot [10] showed existence locally in time. Global existence of solutions which start off close to absolute equilibrium has been proved by Nishida and Imai [12] and Shizuta and Asano [14]. The asymptotic equivalence of the Boltzmann and the gas dynamic equations was demonstrated by Grad [8] and Nishida [13] for initial data near to global equilibrium.

The basis for the present work is the paper by Caflisch and Papanicolaou [2] on the Broadwell model of the Boltzmann equation. The main difficulty in extending those results is the treatment of the high velocity tail of the distribution, as described in Section 4. The key to both these papers is the fact that, after assuming that the nonlinear fluid equations can be solved, the remaining problem is only weakly nonlinear.

In Section 2, the existence theorem is stated. Its proof occupies the rest of the paper. The solution is found as a Hilbert expansion with remainder, as described in Section 3, and the equation for the remainder term is decomposed into low and high velocity components in Section 4. This uses a global Maxwellian distribution ω_M, as well as the local Maxwellian ω associated with the Euler equations. Linear and nonlinear collision operators involving these two Maxwellians are defined in Section 4. Basic estimates for these operators are presented in Section 5. Then a linearized version of the decomposed remainder equations is analyzed in Section 6. Finally, the nonlinear equations are solved by iteration in Section 7.

Use of the Hilbert expansion in an existence theorem for the Boltzmann equation was suggested by George C. Papanicolaou. I also want to thank Harold Grad for a number of discussions. Most of this work was completed at the Mathematics Research Center and the Courant Institute of Mathematical Sciences, whose support I am happy to acknowledge.

2. The Existence Theorem

The Boltzmann equation is

$$\left(\frac{\partial}{\partial t}+\xi_1\nabla\right)F = \frac{1}{\varepsilon}\,Q(F,F) \tag{2.1}$$

in which $\nabla = \partial/\partial x$. The function $F = F(x, t, \boldsymbol{\xi})$ is the density of particles of velocity $\boldsymbol{\xi} \in R^3$, at position $x \in [0, 1]$ and time $t \in [0, \infty)$. We look for solutions which are spatially periodic in $[0, 1]$. The left-hand side of (2.1) represents streaming, while Q is a nonlinear integral operator representing collisions. Following Grad [6], we consider cut-off, hard potentials only. The parameter ε is proportional to the mean free time and is assumed to be small.

There are special distribution functions ω, called Maxwellians and given by

$$\omega(\boldsymbol{\xi}) = \frac{\rho}{(2\pi T)^{3/2}} \exp\{-(\boldsymbol{\xi} - \mathbf{u})^2/2T\}, \tag{2.2}$$

which are in equilibrium with the collision process, i.e.,

$$Q(\omega, \omega) = 0. \tag{2.3}$$

If ρ, $\mathbf{u}$, T *are constant in* x *and* t (as well as in $\boldsymbol{\xi}$), ω is called a *global Maxwellian*: if they depend on x and t, it is a *local Maxwellian*. The constants ρ, $\mathbf{u}$, T are the macroscopic density, velocity and temperature, respectively. More explicit descriptions as well as derivations and basic properties of the Boltzmann equation can be found in [5].

The fluid dynamic description of a gas is given by the Euler equations

$$\begin{aligned} &\frac{\partial}{\partial t}\rho + \nabla\rho u_1 = 0, \\ &\frac{\partial}{\partial t}\rho\mathbf{u} + \nabla(\rho u_1 \mathbf{u}) + \nabla p = 0, \\ &\frac{\partial}{\partial t}\rho(e + \tfrac{1}{2}u^2) + \nabla(\rho u_1(e + \tfrac{1}{2}u^2)) + \nabla(p u_1) = 0, \\ &p = \rho R T = \tfrac{2}{3}\rho e. \end{aligned} \tag{2.4}$$

These are the equations of conservation of mass, momentum, and energy, and the equation of state.

The following theorem shows that there are solutions of the Boltzmann equation which are nearly local Maxwellian, in which the macroscopic variables ρ, $\mathbf{u}$, T evolve according to the Euler equations.

THEOREM. *Let* $(\rho(x, t), \mathbf{u}(x, t), T(x, t))$ *be a smooth, spatially periodic solution of the Euler equations* (2.4) *for* $t \in [0, \tau]$, $x \in [0, 1]$. *Construct the local Maxwellian* $\omega(x, t, \boldsymbol{\xi})$ *from* $\rho, \mathbf{u}, T$ *as in* (2.2). *There is a positive* ε_0 *such that, for each* $0 < \varepsilon \leqq \varepsilon_0$, *a smooth solution* F^ε *of the Boltzmann equation* (2.1) *exists for* $t \in [0, \tau]$, *with*

$$\begin{aligned} &F^\varepsilon \in L^\infty([0, \tau]; H^1(x, \xi)) \cap C([0, T]; L^2), \\ &\frac{d}{dt}F^\varepsilon \in L^\infty([0, \tau]; L^2). \end{aligned} \tag{2.5}$$

Moreover,

$$\|F^\varepsilon - \omega\| \leqq c\varepsilon\,, \tag{2.6}$$

where the norm is in any of the spaces in (2.5) *and c is independent of* ε.

$L^\infty([0,\tau]; L^2)$ means the Banach space of bounded measurable functions from $[0,\tau]$ to $L^2([0,1], R^3)$, etc.

Remarks. (1) This theorem assumes a solution of the nonlinear Euler equations is in hand. Although these equations are much simpler than the Boltzmann equation, the existence question has been only partially answered.

The smoothness of the Euler equations is expected to last until the first appearance of a shock, which we would take as the time τ. We have not optimized the amount of smoothness needed, but certainly it is enough for ρ, $\mathbf{u}$, T to be in H^9.

(2). The initial values of F^ε are essentially those of the Hilbert expansion (see Section 3). This is just what is needed to match to an initial layer.

(3) The one-dimensional spatially periodic problem is handled for simplicity. We could just as well do the multi-dimensional infinite space problem, which is needed for the matching to boundary layers.

(4) The Navier–Stokes equations could be used instead of the Euler equations. The viscosity is multiplied by ε and uniform smoothness as well as nice limiting behavior is required of the solutions. The result is an approximation with error size ε^2.

3. The Hilbert Expansion with Remainder

The solution F^ε is found as a sum

$$F^\varepsilon = \sum_{n=0}^{6} \varepsilon^n F_n + \varepsilon^3 F_R^\varepsilon\,, \tag{3.1}$$

where $F_0, \cdots, F_6$ are independent of ε. They are the first 6 terms of the Hilbert expansion, which solve the equations

$$0 = Q(F_0, F_0)\,, \tag{3.2}$$

$$\left(\frac{\partial}{\partial t} + \xi_1 \nabla\right) F_0 = 2Q(F_0, F_1)\,, \tag{3.3}$$

$$\left(\frac{\partial}{\partial t} + \xi_1 \nabla\right) F_1 = 2Q(F_0, F_2) + Q(F_1, F_1)\,, \tag{3.4}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\left(\frac{\partial}{\partial t} + \xi_1 \nabla\right) F_5 = 2Q(F_0, F_6) + \sum_{\substack{i+j=6\\ 1\leqq i\leqq 5\\ 1\leqq j\leqq 5}} Q(F_i, F_j)\,. \tag{3.5}$$

From equation (3.2) we infer that $F_0=\omega$, a local Maxwellian. The remaining equations (3.3)–(3.5) involve the linear operator $\mathscr{L}=-2Q(F_0,\cdot)$, an integral operator over the velocity space ($\boldsymbol{\xi}$). This Fredholm operator can be inverted after checking that the inhomogeneity is perpendicular to the null space $\{\phi_0,\cdots,\phi_5\}$ of the adjoint operator $\mathscr{L}\dagger$. The appropriate inner product is

$$\langle f, g\rangle=\int_{\mathbf{R}^3} f(\boldsymbol{\xi})g(\boldsymbol{\xi})\,d\boldsymbol{\xi},$$

and the ϕ_i are given by

$$\begin{aligned} \phi_0&=\frac{1}{\rho}, \\ \phi_i&=\frac{1}{\rho T}(\boldsymbol{\xi}_i-\mathbf{u}_i), \\ \phi_4&=\frac{1}{6\rho T^2}((\boldsymbol{\xi}-\mathbf{u})^2-3T), \end{aligned} \tag{3.6}$$

so that $\langle\omega\phi_i,\phi_j\rangle=\delta_{ij}$. The null space of $\mathscr{L}$ is $\{\omega\phi_0,\cdots,\omega\phi_5\}$.

Equations (3.3) and (3.4) become

$$\left\langle\phi_i,\left(\frac{\partial}{\partial t}+\xi_1\frac{\partial}{\partial x}\right)F_0\right\rangle=0, \tag{3.7}$$

$$F_1=-\mathscr{L}^{-1}\left(\left(\frac{\partial}{\partial t}+\xi_1\frac{\partial}{\partial x}\right)F_0\right)+\Phi_1, \tag{3.8}$$

$$\left\langle\phi_i,\left(\frac{\partial}{\partial t}+\xi_1\frac{\partial}{\partial x}\right)F_1-Q(F_1,F_1)\right\rangle=0, \tag{3.9}$$

$$F_2=-\mathscr{L}^{-1}\left(\left(\frac{\partial}{\partial t}+\xi_1\nabla\right)F_1-Q(F_1,F_1)\right)+\Phi_2, \tag{3.10}$$

in which $\mathscr{L}\Phi_1=\mathscr{L}\Phi_2=0$. Equation (3.7) gives exactly the nonliner Euler equations (2.4) for the macroscopic density, velocity, and temperature ρ, $\mathbf{u}$, T corresponding to $\omega=F_0$. From equation (3.8), we see that $F_1=(\rho^1\phi_0+u_i^1\phi_i+T^1\phi_4)\omega+\Psi_1$ in which $\mathscr{L}\Psi_1=-(\partial/\partial t+\xi_1\nabla)F_0$ and $\Psi_1\in N(\mathscr{L})^\perp$. The coefficients ρ^1, $\mathbf{u}^1$, T^1 are the macroscopic density, velocity and temperature corresponding to F_1. According to (3.9) they satisfy inhomogeneous Euler equations linearized about ρ, $\mathbf{u}$, T and with inhomogeneity given as an operator on ρ, $\mathbf{u}$, T. The terms $F_2,\cdots,F_6$ are found similarly. However since the expansion is truncated we can take $\rho^6=\mathbf{u}^6=T^6=0$.

A careful treatment of the Hilbert expansion is found in [5] and [7]. Only

several facts are needed here. We are starting with a smooth solution $(\rho, \mathbf{u}, T)$ of the nonlinear Euler equations, from which we construct $F_0 = \omega$ which solves (3.7). The remaining Hilbert expansion equations are linear and have solutions which are smooth in (x, t) and decay in $\boldsymbol{\xi}$. Consider F_1. Grad has shown in [7] that $\mathscr{L}^{-1}$ preserves decay in $\boldsymbol{\xi}$, so that $\Psi_1 \sim |\boldsymbol{\xi}|^3 \exp\{-(\boldsymbol{\xi}-\mathbf{u})^2/2T\}$. Since the inversion is local in (x, t), Ψ_1 is smooth in (x, t). The remaining terms in F_1 obviously decay like $|\xi|^2 \exp\{-(\boldsymbol{\xi}-\mathbf{u})^2/2T\}$, and they are smooth in (x, t) since coefficients ρ^1, $\mathbf{u}^1$, T^1 solve linear equations with forcing terms coming from the smooth functions ρ, $\mathbf{u}$, T. The following proposition summarizes these facts.

Proposition 3.1. *Let $(\rho, \mathbf{u}, T)$ be a smooth solution of the Euler equations (2.4), and form the Maxwellian $F_0 = \omega$ as in (2.2). Then the terms $F_1, \cdots, F_6$ of the Hilbert expansion are smooth in (x, t) and have decay given by*

$$F_i(x, t, \boldsymbol{\xi}) \leqq c\,|\xi|^{3i}\omega(x, t, \boldsymbol{\xi})\,, \tag{3.11}$$

where c is a constant independent of $\boldsymbol{\xi}$, x, t.

Next we find an equation for the remainder F_R^ε, by putting the expansion (3.1) into equation (2.1) and subtracting the Hilbert expansion equations (3.2)–(3.5). After dividing by ε^3 and regrouping, the equation becomes

$$\left(\frac{\partial}{\partial t} + \xi_1 \nabla\right) F_R = \frac{2}{\varepsilon^2} Q(\omega, F_R) + 2Q(F_1 + \varepsilon F_2 + \varepsilon^2 F_3, F_R) + \varepsilon^2 Q(F_R, F_R) + \varepsilon^2 A\,, \tag{3.12}$$

where

$$A = -\left(\frac{\partial}{\partial t} + \xi_1 \nabla\right) F_5 + \sum_{\substack{i+j \geq 6 \\ i \leq 5,\, j \leq 5}} \varepsilon^{i+j-6} Q(F_1, F_j)\,. \tag{3.13}$$

For initial values we take

$$F_R(t=0) = 0\,. \tag{3.14}$$

The superscript ε on F_R has been dropped. Once the remainder equation (3.12), (3.14) is solved, we have the desired solution of the Boltzmann equation. By writing the solution as in (3.1) we have reduced the Boltzmann equation to an equation in which the nonlinearity and inhomogeneity are both small.

4. Decomposition of F_R

In the past the linearized Boltzmann equation has been analyzed by symmetrizing the operator $\mathscr{L} = -2Q(\omega, \cdot)$ to get $L = -2\omega^{-1/2}Q(\omega, \omega^{-1/2}\cdot)$. Since L is non-negative the $(1/\varepsilon)L$ term does not cause growth. This is fine if ω is constant in (x, t), but otherwise the symmetrization procedure results in a new term $\{\omega^{-1/2}(\partial/\partial t + \xi_1 \partial/\partial x)(\omega^{1/2})\}f$ which is like $|\xi|^3 f$ and has an uncontrolled sign. Thus at large velocities, the distribution function may be growing rapidly due to streaming. Similar problems occur in the study of soft potentials (cf. [3]), shocks (cf. [11]), and boundaries.

To remedy this difficulty, we decompose F_R into essentially low velocity and high velocity parts using two different Maxwellians—ω, as above and

(4.1) $$\omega_M = \frac{1}{(2\pi T_M)^{1/2}} \exp\{-\xi^2/2T_m\},$$

in which T_M is constant, and $T_M > \max_{x,t} T(x, t)$ so that $\omega_M \geqq c\omega$ for all (x, t, ξ). We shall employ various operators which are linearized about these Maxwellians.

Notation.

(4.2) $$Lf = -2\omega^{-1/2}Q(\omega, \omega^{1/2}f) = (\nu + K)f,$$

where ν is a function of ξ and K is a compact integral operator over ξ,

(4.3) $$\nu(\xi) = \int \omega . d\Omega,$$

(4.4) $$\nu\Gamma(f, g) = \omega^{-1/2}Q(\omega^{1/2}f, \omega^{1/2}g).$$

The definitions above are those of Grad [7], [8] (the cross section $d\Omega$ is defined there). The following are new, but are analogous to Grad's definitions:

(4.5) $$L_M f = -2\omega_M^{-1/2}Q(\omega, \omega_M^{1/2}f) = (\nu + K_M)f,$$

(4.6) $$L_1^\varepsilon f = 2\omega_M^{-1/2}Q(F_1 + \varepsilon F_2 + \varepsilon^2 F_3, \omega_M^{1/2}f),$$

(4.7) $$\nu_M \Gamma_M(f, g) = \omega_M^{-1/2}Q(\omega_M^{1/2}f, \omega_M^{1/2}g),$$

(4.8) $$\nu_M(\xi) = \int \omega_M \, d\Omega.$$

We also define

$$\chi(\xi)=\begin{cases}1 & \text{for } |\xi|\leq\xi_0,\\ 0 & \text{for } |\xi|>\xi_0,\end{cases} \tag{4.9}$$

$$\bar{\chi}=1-\chi, \tag{4.10}$$

$$\sigma=(\omega/\omega_M)^{1/2}, \tag{4.11}$$

$$\mu=\tfrac{1}{2}\omega^{-1}\left(\frac{\partial}{\partial t}+\xi_1\nabla\right)\omega, \tag{4.12}$$

$$a=\omega_M^{-1/2}A. \tag{4.13}$$

Note that σ is exponentially decaying since $T_M>T$ for all x, t.

Now decompose F_R as

$$F_R=\omega^{1/2}g+\omega_M^{1/2}h. \tag{4.14}$$

Essentially, $\omega^{1/2}g$ is the low velocity part and $\omega_M^{1/2}h$ the high velocity part; the precise definitions are that g and h solve

$$\left(\frac{\partial}{\partial t}+\xi_1\nabla\right)g=-\frac{1}{\varepsilon}Lg-\frac{1}{\varepsilon}\chi\sigma^{-1}K_Mh, \tag{4.15}$$

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\xi_1\nabla\right)h=&-\mu\sigma g-\frac{1}{\varepsilon}(\nu+\bar{\chi}K_M)h+L_1(\sigma g+h)\\ &+\varepsilon^2\nu_M\Gamma_M(\sigma g+h,\sigma g+h)+\varepsilon^2 a,\end{aligned} \tag{4.16}$$

$$g(t=0)=h(t=0)=0. \tag{4.17}$$

If (4.15) is multiplied by $\omega^{1/2}$ and (4.16) by $\omega_M^{1/2}$, and the two are added, the result is just equation (3.12) for F_R defined by (4.14). After solving for g and h, the solution F of the Boltzmann equation (2.1) will be complete.

5. Basic Estimates

Before analyzing equations (4.15) and (4.16), basic estimates are needed for the operators defined in the last section. First we define some norms.

Definitions.

$$\|f\|_r=\sup_{0\leq t\leq\tau}\sup_{\xi}(1+|\xi|)^r\left(\int f(x,\xi,t)^2\,dx\right)^{1/2}, \tag{5.1}$$

$$\|f\|_{r,s}=\sum_{n=0}^{s}\|\nabla^n f\|_r, \tag{5.2}$$

$$|||f(t)||| = \left(\int f(\mathbf{x}, \boldsymbol{\xi}, t)^2 \, d\boldsymbol{\xi} \, d\mathbf{x} \right)^{1/2}, \tag{5.3}$$

$$|||f(t)|||_s \leq \sum_{n=0}^{s} |||\nabla^n f(t)|||. \tag{5.4}$$

In the estimates of this paper we use c as a generic constant. Any other constant, e.g., c^2+1, will be replaced by c. Often all constants will be omitted.

Basic Estimates. Grad [7], [8] proved the following estimates:

$$\|Kf\|_r \leq \|f\|_{r-1}, \tag{5.5}$$

$$\|Kf\|_0 \leq \sup_{0 \leq t \leq T} |||f|||, \tag{5.6}$$

$$|||Kf||| \leq |||f||| \leq \|f\|_2, \tag{5.7}$$

$$(Lf, f) \geq 0, \tag{5.8}$$

$$\|\Gamma(f, g)\|_r \leq \|f\|_{r,1} \|g\|_r, \qquad r \geq 1, \tag{5.9}$$

$$c \leq \nu(\boldsymbol{\xi}) \leq |\boldsymbol{\xi}|. \tag{5.10}$$

There are analogous estimates for the operators K_M and Γ_M:

$$\|K_M f\|_r \leq \|f\|_{r-1}, \tag{5.11}$$

$$\|K_M f\|_0 \leq \sup_{0 \leq t \leq T} |||f|||, \tag{5.12}$$

$$|||K_M f||| \leq |||f||| \leq \|f\|_2, \tag{5.13}$$

$$\|\Gamma_M(f, g)\|_r \leq \|f\|_{r,1} \|g\|_r, \qquad r \geq 1. \tag{5.14}$$

We also need the following new estimates:

$$\left\| \frac{1}{\nu} L_1 f \right\|_r \leq \|f\|_r, \tag{5.15}$$

$$\left\| \frac{1}{\nu} (\nabla L_M) f \right\|_r \leq \|f\|_r, \tag{5.16}$$

$$\left\| \frac{1}{\nu} (\nabla L_1) f \right\|_r \leq \|f\|_r, \tag{5.17}$$

$$\left\|\frac{1}{\nu}(\nabla K)f\right\|_r \leqq \|f\|_{r+2}, \tag{5.18}$$

$$\nu_M \leqq c\nu, \tag{5.19}$$

$$\nabla\nu \leqq c\nu. \tag{5.20}$$

The integral operator K_M is written explicitly, as in the analysis of Grad [7], in the form

$$K_M f(\xi) = \int_{R^3} l(\xi, \eta) f(\eta)\, d\eta,$$

$$l(\xi,\eta) = -l_1(\xi,\eta) + l_2(\xi,\eta),$$

$$l_1(\xi,\eta) = 2\pi \frac{\omega(\xi)}{\omega_M(\xi)^{1/2}} \omega_M^{1/2}(\eta) \int_{R^3} B(\theta, \mathbf{v})\, d\theta, \tag{5.21}$$

$$l_2(\xi,\eta) = \frac{2}{(2\pi)^{3/2}} \frac{1}{v^2} \exp\left\{-\frac{1}{2T}\left(\zeta_1^2 + \tfrac{1}{4}v^2 - \left(1 - \frac{T}{T_M}\right)\zeta_1 \cdot \mathbf{v} + \frac{T}{T_M}\mathbf{u}_1 \cdot \mathbf{v}\right)\right\}$$

$$\times \int_{\mathbf{w}\perp\mathbf{v}} \exp\left\{-\frac{1}{2T}(\zeta_2 + \mathbf{w})^2\right\} Q(\mathbf{v}, \mathbf{w})\, d\mathbf{w},$$

where $v = \eta - \xi$, $\zeta = \frac{1}{2}(\xi + \eta) = \zeta_1 + \zeta_2$, with u_1 the component of $\mathbf{u}$ parallel to $\mathbf{v}$, ζ_1 the component of ζ parallel to $\mathbf{v}$. The estimates (5.11), (5.12), (5.13) are proved using the bound

$$|l(\xi,\eta)| \leqq c\frac{1}{v}\exp\{-c(v^2 + \zeta_1^2)\}. \tag{5.22}$$

The estimate (5.14) is exactly (5.9) with ω replaced by ω_M. The inequalities for ν, ν_M, $\nabla\nu$ are derived directly from the definitions (4.3), (4.8). Bounds on L_1, ∇L_1, ∇L_M, ∇K are proved using (5.9) or (5.14), (5.19), and (5.20), and the smoothness and decay of the F_i.

The next lemma uses the basic estimates, the exponential decay of σ, and the smoothness and decay of the F_i.

LEMMA 5.1. *The following estimates hold:*

$$\sup_{\xi} (1 + |\xi|)^n \sigma(\xi) \leqq c, \tag{5.23}$$

$$\|\sigma g\|_{r,s} \leqq c\,\|g\|_{0,s}\,, \tag{5.24}$$

$$\left\|\frac{1}{\nu}\,\nu_M\Gamma_M(\sigma g_1+h_1,\,\sigma g_2+h_2)\right\|_{r,s} \leqq (\|g_1\|_{0,s}+\|h_1\|_{r,s})(\|g_2\|_{0,s}+\|h_2\|_{r,s})\,, \tag{5.25}$$

$$r\geqq 1\,,\quad s\geqq 1\,,$$

$$\left\|\frac{1}{\nu}\,a\right\|_{r,s} \leqq c\,. \tag{5.26}$$

In Section 6, the following modified Gronwall inequality is used.

LEMMA 5.2. *Let* $u(t)\geqq 0$, $\alpha\geqq 0$, $\beta\geqq 0$, *and suppose that*

$$\frac{d}{dt}\,u(t)\leqq \alpha \sup_{0\leqq s\leqq t} u(s)+\beta(t)\,. \tag{5.27}$$

Then

$$u(t)\leqq u(0)e^{\alpha t}+\int_0^t e^{\alpha(t-s)}\beta(s)\,ds\,. \tag{5.28}$$

6. The Linearized Equation

The linearized version of equations (4.15) and (4.16) is:

$$\left(\frac{\partial}{\partial t}+\xi_1\nabla\right)g=-\frac{1}{\varepsilon}\,Lg-\frac{1}{\varepsilon}\,\chi\sigma^{-1}K_M h\,, \tag{6.1}$$

$$\left(\frac{\partial}{\partial t}+\xi_1\nabla\right)h=-\mu\sigma g-\frac{1}{\varepsilon}\,(\nu+\bar{\chi}K_M)h+L_1(\sigma g+h)+\varepsilon^2 b\,, \tag{6.2}$$

$$g(t=0)=h(t=0)=0\,. \tag{6.3}$$

The estimates in the next lemma are the key steps in this paper.

LEMMA 6.1. *Let g, h solve equations* (6.1), (6.2), (6.3). *Then*

$$\|g\|_{r,1}\leqq \varepsilon^{1/4}\,\|b/\nu\|_{r+1,s}\,, \tag{6.4}$$

$$\|h\|_{r,1}\leqq \varepsilon^{5/4}\,\|b/\nu\|_{r,s}\,, \tag{6.5}$$

for $r\geqq 3$, $s\geqq 1$.

Proof: (A) **Estimates on h.** The integral version of (6.2) is

$$h(x,t,\xi)=\int_0^t \exp\{-(t-s)\nu(\xi)/\varepsilon\}\left\{-\mu\sigma g-\frac{1}{\varepsilon}\bar{\chi}K_M h+L_1(\sigma g+h)+\varepsilon^2 b\right\} \times(x-(t-s)\xi_1,\xi,s)\,ds. \tag{6.6}$$

Using Grad's basic lemma, [8], p. 164, we estimate

$$\begin{aligned}\|h\|_r &\leqq \varepsilon\left\|\frac{1}{\nu}\mu\sigma g\right\|_r+\left\|\frac{1}{\nu}\bar{\chi}K_M h\right\|_r+\varepsilon\left\|\frac{1}{\nu}L_1(\sigma g+h)\right\|_r+\varepsilon^3\left\|\frac{1}{\nu}b\right\|_r \\ &\leqq \varepsilon\|g\|_0+\frac{1}{c_0}\|h\|_r+\varepsilon(\|g\|_0+\|h\|_r)+\varepsilon^3\|b/\nu\|_r,\end{aligned} \tag{6.7}$$

making use of (5.10), (5.11), (5.15), (5.24), (4.10). By choosing ξ_0 large enough, it follows that

$$\|h\|_r \leqq \varepsilon\|g\|_0+\varepsilon^3\|b/\nu\|_r. \tag{6.8}$$

We differentiate (6.2) to get an equation for ∇h:

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\xi_1\nabla\right)\nabla h=&-(\mu\sigma\nabla g+(\nabla\mu\sigma)g)-\frac{1}{\varepsilon}(\nu+\bar{\chi}K_M)\nabla h-\frac{1}{\varepsilon}(\nabla\nu+\bar{\chi}K_M)h\\ &+L_1(\sigma\nabla g+\nabla h+(\nabla\sigma)g)+(\nabla L_1)(\sigma g+h)+\varepsilon^2\nabla b,\end{aligned} \tag{6.9}$$

and continue as before to obtain

$$\|\nabla h\|_r \leqq \varepsilon\|g\|_{0,1}+\varepsilon^3\|b/\nu\|_r+\varepsilon^3\left\|\frac{1}{\nu}\nabla b\right\|_r. \tag{6.10}$$

Combined with (6.8), this says that

$$\|h\|_{r,1} \leqq \varepsilon\|g\|_{0,1}+\varepsilon^3\|b/\nu\|_{r,1}. \tag{6.11}$$

(B) **Estimates on g.** The integral version of (6.1) is

$$\begin{aligned}&g(x,t,\xi)\\ &=\int_0^t \exp\{-(t-s)\nu(\xi)/\varepsilon\}\left\{-\frac{1}{\varepsilon}Kg-\frac{1}{\varepsilon}\chi\sigma^{-1}K_M h\right\}(x-(t-s)\xi_1,\xi,s)\,ds.\end{aligned} \tag{6.12}$$

Making use of Grad's lemma we find

$$\|g\|_r \leqq \|Kg\|_r + \|\chi\sigma^{-1}K_M h\|_r . \tag{6.13}$$

But

$$\begin{aligned} \|\sigma^{-1}\chi K_M h\|_r &\leqq \exp\{c\xi_0^2\}\,\|K_M h\|_r \\ &\leqq \exp\{c\xi_0^2\}(\varepsilon\,\|g\|_0 + \varepsilon^3\,\|b/\nu\|_{r-1}) \\ &\leqq \varepsilon^{1/2}\,\|g\|_0 + \varepsilon^{5/2}\,\|b/\nu\|_{r-1}, \end{aligned} \tag{6.14}$$

using (5.11) and (6.8) and choosing ε small enough so that $\varepsilon \exp\{c\xi_0^2\} < \varepsilon^{1/2}$. The first term in (6.13) is estimated by (5.5) or (5.6), with the result that

$$\|g\|_r \leqq \|g\|_{r-1} + \varepsilon^{5/2}\,\|b/\nu\|_{r-1}, \qquad r \geqq 1, \tag{6.15}$$

$$\|g\|_0 \leqq \sup_{0 \leqq t \leqq \tau} |||g||| + \varepsilon^{5/2}\,\|b/\nu\|_0 . \tag{6.16}$$

Employing (6.15) for r and then for $(r-1)$, recursively, and finally using (6.16), yields

$$\|g\|_r \leqq \sup_{0 \leqq t \leqq \tau} |||g||| + \varepsilon^{5/2}\,\|b/\nu\|_{r-1} . \tag{6.17}$$

To estimate $|||g|||$, we multiply equation (6.1) by g and integrate over x and ξ. We use the spatial periodicity and the non-negativity of L to drop the $\partial/\partial x$ and the L terms. the result is

$$\frac{1}{2}\frac{\partial}{\partial t}|||g(t)|||^2 \leqq \frac{1}{\varepsilon}|||\sigma^{-1}\chi K_M h||| \cdot |||g||| . \tag{6.18}$$

Thus

$$\begin{aligned} \frac{\partial}{\partial t}|||g(t)||| &\leqq \exp\{c\xi_0^2\}\frac{1}{\varepsilon}\|h\|_2 \\ &\leqq \exp\{c\xi_0^2\}(\|g\|_0 + \varepsilon^2\,\|b/\nu\|_2) \\ &\leqq \exp\{c\xi_0^2\}\Big(\sup_{0 \leqq s \leqq \tau} |||g(s)||| + \varepsilon^2\,\|b/\nu\|_2\Big). \end{aligned} \tag{6.19}$$

Since this is true for any τ, $\tau \geqq t$, the modified Gronwall inequality is

applicable and implies that

$$\begin{aligned}(6.20)\qquad |||g(t)||| &\leq \varepsilon^2 \|b/\nu\|_2 \exp\{t\exp\{c\xi_0^2\}\} \\ &\leq \varepsilon^{3/2}\|b/\nu\|,\end{aligned}$$

by choosing ε small enough so that $\sqrt{\varepsilon}\exp\{\tau\exp\{c\xi_0^2\}\} \leq 1$. Substitution into (6.17), gives the result

$$(6.21)\qquad \|g\|_r \leq \varepsilon^{3/2}\|b/\nu\|_{r-1}, \qquad r \geq 3.$$

(C) **Estimates on ∇g.** The equation for ∇g is

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\xi_1\nabla\right)\nabla g &= -\frac{1}{\varepsilon}(\nu+K)\nabla g - \frac{1}{\varepsilon}(\nabla\nu+\nabla K)g \\ (6.22)\qquad &-\frac{1}{\varepsilon}\chi(\nabla(\sigma^{-1})K_M+\sigma^{-1}(\nabla K_M))h - \frac{1}{\varepsilon}\chi\sigma^{-1}K_M(\nabla h),\end{aligned}$$

and proceeding as before we find

$$(6.23)\qquad \|\nabla g\|_r \leq \sup_{0\leq t\leq \tau} |||\nabla g||| + \varepsilon^{3/2}\|b/\nu\|_{r+1,1}.$$

The L^2 estimate on ∇g is found by multiplying equation (6.22) by ∇g and integrating over x and ξ. As before, the x derivative and L, g terms can be omitted. Hence

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}|||\nabla g|||^2 &\leq -\frac{1}{\varepsilon}((\nabla\nu+\nabla K)g,\nabla g) \\ (6.24)\qquad &-\frac{1}{\varepsilon}(\chi(\nabla(\sigma^{-1})K_M+\sigma^{-1}(\nabla K_M))h,\nabla g) \\ &-\frac{1}{\varepsilon}(\chi\sigma^{-1}K_M(\nabla h),\nabla g).\end{aligned}$$

We use the Schwartz inequality to estimate

$$\begin{aligned}\frac{\partial}{\partial t}|||\nabla g||| &\leq \frac{1}{\varepsilon}|||(\nabla\nu+\nabla K)g||| + \frac{1}{\varepsilon}|||\chi(\nabla(\sigma^{-1})K_M+\sigma^{-1}(\nabla K_M))h||| \\ (6.25)\qquad &+\frac{1}{\varepsilon}|||\chi\sigma^{-1}K_M\nabla h||| \leq \frac{1}{\varepsilon}\|g\|_4 + \frac{1}{\varepsilon}\exp\{c\xi_0^2\}\|h\|_{2,1} \\ &\leq \varepsilon^{1/2}\|b/\nu\|_3 + \exp\{c\xi_0^2\}(\|g\|_{0,1}+\varepsilon^2\|b/\nu\|_{2,1}) \\ &\leq \exp\{c\xi_0^2\}\sup_{0\leq t\leq \tau}|||\nabla g||| + \varepsilon^{1/2}\|b/\nu\|_{3,1}.\end{aligned}$$

From the Gronwall inequality we obtain

$$\text{(6.26)} \qquad |||\nabla g|||(t) \leqq \varepsilon^{1/2} \|b/\nu\|_{3,1} \exp\{\exp\{c\xi_0^2\}\} \leqq \varepsilon^{1/4} \|b/\nu\|_{3,1},$$

by choosing ε sufficiently small. Combining this with (6.23), (6.21), (6.8), and (6.11), we get

$$\text{(6.27)} \qquad \begin{aligned} &\|g\|_{r,1} \leqq \varepsilon^{1/4} \|b/\nu\|_{r+1,1}, \qquad && r \geqq 2, \\ &\|h\|_{r,1} \leqq \varepsilon^{5/4} \|b/\nu\|_{r,1}, && r \geqq 3. \end{aligned}$$

The same estimates are true with $s > 1$; thus the proof of Lemma 6.1 is complete.

7. Solution of the Nonlinear Equations

The nonlinear equations (4.15), (4.16), (4.17) are solved by the implicit function theorem or direct iteration on the linear equations. Let g_{n+1}, h_{n+1} be the solution of equations (6.1), (6.2), (6.3), with

$$b = \nu_M \Gamma_M(\sigma g_n + h_n, \sigma g_n + h_n) + a,$$

and $g_0 = h_0 = 0$. Then

$$\text{(7.1)} \qquad \left\| \frac{1}{\nu} b \right\|_{r,1} \leqq \|g_n\|_{0,1}^2 + \|h_n\|_{r,1}^2 + c,$$

and from Lemma 6.1, we get

$$\text{(7.2)} \qquad \begin{aligned} \|g_{n+1}\|_{r,1} &\leqq \varepsilon^{1/4}(\|g_n\|_{0,1}^2 + \|h_n\|_{r,1}^2 + c), \\ \|h_{n+1}\|_{r,1} &\leqq \varepsilon^{1/4}(\|g_n\|_{0,1}^2 + \|h_n\|_{r,1}^2 + c). \end{aligned}$$

By iteration we find

$$\text{(7.3)} \qquad \|g_{n+1}\|_{r,1} \leqq c, \qquad \|h_{n+1}\|_{r,1} \leqq c,$$

in which c is independent of n. The differences $g_{n+1} - g_n$, $h_{n+1} - h_n$ solve equations (6.1), (6.2), (6.3) with

$$\text{(7.4)} \qquad \begin{aligned} &b = \nu_m \Gamma_M(\sigma g_n + h_n + \sigma g_{n-1} + h_{n-1}, \sigma g_n + h_n - \sigma g_{n-1} - h_{n-1}), \\ &\left\| \frac{1}{\nu} b \right\|_{r,1} \leqq c(\|g_n - g_{n-1}\|_{0,1} + \|h_n - h_{n-1}\|_{r,1}), \end{aligned}$$

so that

$$\begin{aligned} \|g_{n+1}-g_n\|_{r,1} &\leqq \varepsilon^{1/4}c(\|g_n-g_{n-1}\|_{0,1}+\|h_n-h_{n-1}\|_{r,1}), \\ \|h_{n+1}-h_n\|_{r,1} &\leqq \varepsilon^{1/4}c(\|g_n-g_{n-1}\|_{0,1}+\|h_n-h_{n-1}\|_{r,1}). \end{aligned} \tag{7.5}$$

From this it follows that $g_n \to g$, $h_n \to h$ in $\|\cdot\|_{r,1}$. By standard arguments one then finds that g, h are solutions of (4.15) and (4.16). This completes the proof of Theorem 2.1.

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SUPPLEMENTARY

INFORMATION

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ERRATA: "The Fluid Dynamic Limit of the Nonlinear Boltzmann Equation" by Russel E. Caflisch
Comm. Pure Appl. Math., 33 (1980), pp. 651-666.

Replace line (2.5) by

$$F^{\varepsilon} \in B_{31} , \qquad \frac{d}{dt} F^{\varepsilon} \in B_{30} ,$$

AD-A091475

in which $B_{rs} = \{f: \ \|f\|_{r,s} \leq \infty\}$ with $\|f\|_{r,s}$ defined by (5.2).